THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050B Mathematical Analysis I (Fall 2016)
Tutorial Questions for 20 Oct

1. (Optional) Show, in two ways, that the following sequences are convergent, and compute their limits.

   (a) $x_1 := 2$, $x_{n+1} := \frac{1}{2}\left(x_n + \frac{1}{x_n}\right)$

   (b) $l := \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}$. In sequence notation, this is to say that $x_1 := \sqrt{2}$, and that $x_{n+1} := \sqrt{x_n + 2}$.

   - (Method 1): Show that the sequences are bounded and monotone, using mathematical induction.
   - (Method 2): Show that the sequences are contractive.

   We would come to this if we had time.

2.

   Theorem 1. *(Non-convergence) Let (x_n) be a sequence of real numbers, and $x \in \mathbb{R}$. Then (x_n) does NOT converge to x if and only if there is $\epsilon_0 > 0$ and a subsequence (x_{n_k}) such that for each $k \in \mathbb{N}$, we have $|x_{n_k} - x| \geq \epsilon_0$.*

3. (Homework 4, Q4)
   Show that $\lim_{n\to\infty} x_n = x \in \mathbb{R}$ if and only if each subsequence (x_{n_k}) of (x_n) has in turn a (sub)subsequence $(x_{n_{k_j}})$ converging to x.

4. (Homework 4, Q5)
   Let (x_n) be a bounded sequence that does not converge to $x \in \mathbb{R}$. Then there is a subsequence (x_{n_k}) converging to some $x' \neq x$.

5. (Difficult, optional) Let (x_n) be a sequence that is both bounded above and below. For $n \in \mathbb{N}$, we denote $s_n := \sup\{x_k : k \geq n\}$ to be the supremum of a tail of (x_n).

   (a) Show that (s_n) is bounded above and below.

   (b) Show that (s_n) is monotonically decreasing.

   (c) Hence, show that (s_n) converges to some limit $s \in \mathbb{R}$. (In standard notations, we denote it as $s := \limsup_{n\to\infty} x_n$).

   (d) Show that there is a subsequence (x_{n_k}) of (x_n) that converges to s.

   (e) It is easily seen that the assumption of (x_n) being bounded above cannot be removed. However, Show that the assumption of (x_n) being bounded below cannot be removed either, by finding an example where (x_n) is bounded only above but not below, such that (s_n) does not converge to any real number.

6. (Limits of Functions)

   By definition of limit of functions, compute the limit of the rational function:

$$\lim_{x\to 0} \frac{x+1}{x-2}$$